

A NOTE ON THE SOME GEOMETRIC PROPERTIES OF THE SEQUENCE SPACES DEFINED BY TAYLOR METHOD

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ABSTRACT. In this paper, it was obtained the new matrix domain with the well known classical sequence spaces and an infinite matrix. The Taylor method which known then as the circle method of order r ($0 < r < 1$), as an infinite matrix for the matrix domain is used. Newly constructed space is isomorphic copy of the spaces of all absolutely p -summable sequences. It is well known that Hilbert space have the nicest geometric properties. Then, it is proved that the new space is a Hilbert space for $p = 2$. Further, it was computed dual spaces and characterized some matrix classes of the new Taylor space in the table form. Section 3 is devoted some geometric properties of Taylor space.

1. INTRODUCTION

The e_p^r and e_∞^r sequence spaces using Euler mean were defined by Altay et al. [1], as follows:

$$e_p^r := \left\{ x = (x_k) \in \omega : \sum_n \left| \sum_{k=0}^n \binom{n}{k} (1-r)^{n-k} r^k x_k \right|^p < \infty \right\}, \quad (1 \leq p < \infty),$$

$$e_\infty^r := \left\{ x = (x_k) \in \omega : \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^n \binom{n}{k} (1-r)^{n-k} r^k x_k \right| < \infty \right\},$$

where $E^r = (e_{nk}^r)$ denotes the Euler means of order r defined by

$$e_{nk}^r = \begin{cases} \binom{n}{k} (1-r)^{n-k} r^k & , \quad (0 \leq k \leq n), \\ 0 & , \quad (k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$. It is known that the method E^r is regular for $0 < r < 1$ and E^r is invertible such that $(E^r)^{-1} = E^{1/r}$ with $r \neq 0$. We assume unless stated otherwise that $0 < r < 1$.

Following Altay et al. [1] and Mursaleen et al.[2], Kirisci [12] defined the sequence spaces t_c^r and t_0^r consisting of all sequences $x = (x_k)$ such that their Taylor transform $T(r)$, as below:

$$t_0^r := \left\{ x = (x_k) \in \omega : \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} \binom{k}{n} (1-r)^{n+1} r^{k-n} x_k = 0 \right\},$$

$$t_c^r := \left\{ x = (x_k) \in \omega : \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} \binom{k}{n} (1-r)^{n+1} r^{k-n} x_k \text{ exists} \right\}.$$

Let $r \in \mathbb{C}/\{0\}$. Then, the Taylor transform $T(r)$ was defined by the matrix $T(r) = (t_{nk}^r)$, where

$$(1.1) \quad t_{nk}^r = \begin{cases} \binom{k}{n} (1-r)^{n+1} r^{k-n} x_k & , \quad (k \geq n), \\ 0 & , \quad (0 \leq k < n) \end{cases}$$

for all $k, n \in \mathbb{N}$. In case $r = 0$, it is immediate that $T(0) = (t_{nk}^r) = (\delta_{nk}) = I$. In working with the Taylor method it is frequently used the fact that

$$\frac{1}{(1-z)^{n+1}} = \sum_{k=n}^{\infty} \binom{k}{n} z^{k-n} \quad \text{if } |z| < 1$$

and additionally $\sum_{k=n}^{\infty} \binom{k}{n} z^{k-n}$ converges only if $|z| < 1$. The Taylor transform $T(r)$ is regular if and only if $0 \leq r < 1$, i.e., r is real and $0 \leq r < 1$. The product of the $T(r)$ matrix with the $T(s)$ matrix is the transpose of the matrix $(1-r)(1-s)E^{(1-r)(1-s)}$, where E denotes Euler mean. It is known that $T(r)$ is invertible and

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$(T(r))^{-1} = T[-r/(1-r)]$ provided $r \neq 1$.

The α -(Köthe Toeplitz), β -(Generalized Köthe Toeplitz) and γ -(Garling)dual spaces of the sequence spaces t_c^r and t_0^r were computed and necessary and sufficient conditions for the charaterization of the matrix classess of $(t_c^r : Y)$, $(t_0^r : Y)$ and $(Y : t_c^r)$, $(Y : t_0^r)$ were given in [12], where Y is one of the classical sequence space. Also, in [12], Steinhaus type theorems were proved.

The aim of this paper is examined some geometric properties of new spaces t_p^r and t_∞^r defined by Taylor transform such as Banach-Saks, weak Banach-Saks, weak fixed point, modulus of convexity.

2. NEW TAYLOR SEQUENCE SPACES OF NON-ABSOLUTE TYPE

In this section, we give the definitions of the sequence spaces t_p^r and t_∞^r and study some properties.

Following Altay et al.[1], Mursaleen et al. [2] and Kirişci [12], we define the sequence spaces t_p^r and t_∞^r , as the set of all sequences such that $T(r)$ -transforms of them are in the spaces ℓ_p and ℓ_∞ , that is

$$\begin{aligned} t_p^r &= \left\{ x = (x_k) \in \omega : \sum_n \left| \sum_{k=n}^{\infty} \binom{k}{n} (1-r)^{n+1} r^{k-n} x_k \right|^p < \infty \right\}, \quad (1 \leq p < \infty) \\ t_\infty^r &= \left\{ x = (x_k) \in \omega : \sup_{n \in \mathbb{N}} \left| \sum_{k=n}^{\infty} \binom{k}{n} (1-r)^{n+1} r^{k-n} x_k \right| < \infty \right\}. \end{aligned}$$

If λ is an arbitrary normed sequence space, then the matrix domain $\lambda_{T(r)}$ is called *Taylor sequence space*. This section of the study is a natural continuation of the results in [12]. So, some topological properties of the spaces t_p^r and t_∞^r are similar to the spaces t_c^r and t_0^r . That is, in point of structural properties, we can obtain similar topological results between new spaces and the spaces t_c^r and t_0^r . Therefore, we won't give the proof of topological properties of new spaces. However, for $p = 2$, we show that the space t_p^r is a Hilbert space. Also, we will prove the theorems of dual spaces which are important.

Define the sequence $y = \{y_k(r)\}$ by the $T(r)$ -transform of a sequence $x = (x_k)$, i.e.,

$$(2.1) \quad y_k(r) = \sum_{n=k}^{\infty} \binom{k}{n} (1-r)^{n+1} r^{k-n} x_k \quad \text{for all } k \in \mathbb{N}.$$

The absolute property does not hold on the spaces t_p^r and t_∞^r , that is $\|x\|_{t_p^r} \neq \|x\|_{t_p^r}$ and $\|x\|_{t_\infty^r} \neq \|x\|_{t_\infty^r}$ for at least one sequence in the spaces t_p^r and t_∞^r , where $|x| = (|x_k|)$. This means that these spaces are sequence spaces of nonabsolute type.

The spaces t_p^r and t_∞^r are linear spaces with coordinatewise addition and scalar multiplication that are *BK*-spaces with the norm $\|x\|_{t_p^r} = \|T(r)x\|_{\ell_p}$, where $1 \leq p \leq \infty$.

The space t_p^r is an isomorphic copy of the classical ℓ_p space, for $1 \leq p \leq \infty$ i.e., $t_p^r \cong \ell_p$ and $t_\infty^r \cong \ell_\infty$.

Theorem 2.1. *Except the case $p = 2$, the space t_p^r is not an inner product space, hence not a Hilbert space for $1 \leq p < \infty$.*

Proof. For $p = 2$, we will show that the space t_2^r is a Hilbert space. It is known that the space t_p^r for $1 \leq p \leq \infty$, then the space t_2^r is a *BK*-space, for $p = 2$. Also its norm can be obtained from an inner product, i.e., $\|x\|_{t_2^r} = \langle T(r)x, T(r)x \rangle^{1/2}$ holds. Then the space t_2^r is a Hilbert space.

If we choose the sequences $e^{(0)} = (1, 0, 0, \dots)$ and $e^{(1)} = (0, 1, 0, \dots)$, we get that the norm of the space t_p^r does not satisfy the parallelogram equality, which means that the norm cannot be obtained from inner product. Hence, the space t_p^r with $p \neq 2$ is a Banach space that is not a Hilbert space. \square

Since the space t_p^r is an isomorphic copy of the classical ℓ_p space, for $1 \leq p \leq \infty$, then, we can consider the transformation S from t_p^r to ℓ_p by $y = Sx = T(r)x$. Clearly, the transformation S is a linear, surjective and norm preserving(From Theorem 2.2 of [12]). Because of the isomorphism S is onto the inverse image of the basis of the space ℓ_p is the basis of the new spaces t_p^r . Therefore, choose a sequence $b^{(k)}(r) = \{b_n^{(k)}(r)\}_{n \in \mathbb{N}}$ of elements of the space t_p^r for every fixed $k \in \mathbb{N}$ by

$$b_n^{(k)}(r) = \begin{cases} \binom{k}{n}(1-r)^{-(k+1)}(-r)^{k-n} & , \quad k \geq n \\ 0 & , \quad 0 \leq k < n \end{cases}$$

and let $\lambda_k(r) = (T(r)x)_k$ for all $k \in \mathbb{N}$. Then, the sequence $\{b^{(k)(r)}\}_{k \in \mathbb{N}}$ is a basis for the space t_p^r , and any $x \in t_p^r$ has a unique representation of the form

$$(2.2) \quad x = \sum_k \lambda_k(r) b^{(k)}(r).$$

Remark 2.2. (i) *It is well known that every Banach space X with Schauder basis is seperable.*
(ii) *Since the sequence space ℓ_∞ has no Schauder basis, Taylor sequeunce space t_∞^r has no Schuder basis. Also, the space t_∞^r is not seperable.*

Corollary 2.3. *The space t_p^r is seperable.*

In following theorem, we give some inclusion relations for the Taylor sequence spaces t_p^r and t_∞^r . These inclusion relations can be proved as similar to inclusion relations theorems in Altay et al.[1], Mursaleen et al. [2] and Kirişçi [12].

Theorem 2.4. *We have:*

- (i) *The inclusions $\ell_p \subset t_p^r$ strictly holds for $1 \leq p < \infty$.*
- (ii) *Neither of the spaces t_p^r and ℓ_∞ includes the other one, where $1 \leq p < \infty$.*
- (iii) *The space t_∞^r strictly includes both the space ℓ_∞ and the spaces t_p^r , where $1 \leq p < \infty$.*
- (iv) *If $0 < s \leq r < 1$, then $t_p^r \subset t_p^s$.*
- (v) *The inclusion $t_p^r \subset t_s^r$ holds whenever $1 \leq p < s$.*

We assume throughtout that $p^{-1} + q^{-1} = 1$ for $p, q \geq 1$ and denote the collection of all finite subsets on \mathbb{N} by \mathcal{F} . Firstly, we give some results for use in proof of the Theorems 2.6-2.8.

$$(2.3) \quad \sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} a_{nk} \right|^q < \infty \quad (1 < p \leq \infty),$$

$$(2.4) \quad \lim_{n \rightarrow \infty} a_{nk} = \alpha_k \quad (k \in \mathbb{N}),$$

$$(2.5) \quad \sup_{n \in \mathbb{N}} \sum_k |a_{nk}|^q < \infty \quad (1 < p < \infty)$$

Lemma 2.5. *For the characterization of the class $(X : Y)$ with $X = \{\ell_p\}$ and $Y = \{\ell_1, c, \ell_\infty\}$, we can give the necessary and sufficient conditions from Table 1, where*

1. (2.3)	2. (2.4), (2.5)	3. (2.5)
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To \rightarrow	ℓ_1	c	ℓ_∞
From \downarrow			
ℓ_p	1.	2.	3.

Table 1

Now, we may give the theorems determining the α -, β - and γ -duals of the Taylor sequence spaces t_p^r for $1 \leq p \leq \infty$.

Theorem 2.6. *The α -duals of the spaces $t_p^r = \alpha_r$ and $t_1^r = \alpha_\infty$, where*

$$\begin{aligned} \alpha_r &= \left\{ a = (a_k) \in \omega : \sup_{N, K \in \mathcal{F}} \sum_{n \in N} \left| \sum_{k \in K} c_{nk}^r \right|^q < \infty \right\} \\ \alpha_\infty &= \left\{ a = (a_k) \in \omega : \sup_{n \in \mathbb{N}} \sum_k |c_{nk}^r| < \infty \right\}. \end{aligned}$$

where the matrix $C(r) = c_{nk}^r$ defined by

$$(2.6) \quad c_{nk}^r = \begin{cases} \binom{k}{n} (-r)^{k-n} (1-r)^{-(k+1)} a_n & , \quad (k \geq n), \\ 0 & , \quad (0 \leq k < n) \end{cases}$$

Proof. We choose the sequence $a = (a_k) \in \omega$. We can easily derive that with the (2.1) that

$$(2.7) \quad a_n x_n = \sum_{k=n}^{\infty} \binom{k}{n} (-r)^{k-n} (1-r)^{-(k+1)} a_n y_k = (C(r)y)_n, \quad (n \in \mathbb{N}).$$

for all $k, n \in \mathbb{N}$, where $C(r) = c_{nk}^r$ defined by (2.6). It follows from (2.7) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in t_p^r$ or t_∞^r if and only if $Cy \in \ell_1$ whenever $y \in \ell_p$ or ℓ_∞ . We obtain that $a \in (t_p^r)^\alpha$ or $a \in (t_\infty^r)^\alpha$ whenever $x \in (t_p^r)$ or $x \in (t_\infty^r)$ if and only if $C \in (\ell_p : \ell_1)$ or $C \in (\ell_\infty : \ell_1)$. Therefore, we get by Lemma 2.5 with the matrix C instead of A that $a \in (t_p^r)^\alpha$ or $a \in (t_\infty^r)^\alpha$ if and only if $\sup_{n \in K} \sum_k |c_{nk}^r| < \infty$. This gives us the result that $(t_p^r)^\alpha = \alpha_r$ and $(t_\infty^r)^\alpha = \alpha_r$. \square

Theorem 2.7. The matrix $D(r) = (d_{nk}^r)$ is defined by

$$(2.8) \quad d_{nk}^r = \begin{cases} \sum_{k=0}^n \binom{n}{k} (-r)^{n-k} (1-r)^{-(n+1)} a_k & , \quad (0 \leq k \leq n), \\ 0 & , \quad (k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$. Then, $(t_1^r)^\beta = \beta_2 \cap \beta_4$, $(t_p^r)^\beta = \beta_1 \cap \beta_2$ and $(t_\infty^r)^\beta = \beta_2 \cap \beta_3$, where

$$\begin{aligned} \beta_1 &= \left\{ a = (a_k) \in \omega : \sup_{n \in \mathbb{N}} \sum_k |d_{nk}^r|^q < \infty \right\} \quad (1 < p < \infty), \\ \beta_2 &= \left\{ a = (a_k) \in \omega : \lim_{n \rightarrow \infty} d_{nk}^r \text{ exists for each } k \in \mathbb{N} \right\}, \\ \beta_3 &= \left\{ a = (a_k) \in \omega : \lim_{n \rightarrow \infty} \sum_k |d_{nk}^r| \text{ exists} \right\}, \\ \beta_4 &= \left\{ a = (a_k) \in \omega : \sup_{k, n \in \mathbb{N}} |d_{nk}^r| < \infty \right\}. \end{aligned}$$

Proof. Consider the equation

$$(2.9) \quad \begin{aligned} \sum_{k=0}^n a_k x_k &= \sum_{k=0}^n \left[\sum_{j=k}^{\infty} \binom{k}{j} (-r)^{k-j} (1-r)^{-(k+1)} y_j \right] a_k \\ &= \sum_{k=0}^n \left[\sum_{j=0}^k \binom{k}{j} (-r)^{k-j} (1-r)^{-(k+1)} a_j \right] y_k = (D(r)y)_n \end{aligned}$$

Then, we deduce from Lemma 2.5 with (2.9) that $ax = (a_k x_k) \in cs$ whenever $x \in t_p^r$ if and only if $D(r)y \in c$ whenever $y \in \ell_p$. That is to say that $a = (a_k) \in (t_p^r)^\beta$ if and only if $D(r) \in (\ell_p : c)$. Therefore, we derive from (2.4) and (2.5), we conclude that $\lim_{n \rightarrow \infty} d_{nk}^r$ exists and $\sup_{n \in \mathbb{N}} \sum_k |d_{nk}^r| < \infty$ which shows that $(t_p^r)^\beta = \beta_1 \cap \beta_2$.

In a similar way, we can prove that $(t_1^r)^\beta = \beta_2 \cap \beta_4$, and $(t_\infty^r)^\beta = \beta_2 \cap \beta_3$. \square

Theorem 2.8. $(t_1^r)^\gamma = \beta_4$, $(t_p^r)^\gamma = \beta_1$ ($1 < p < \infty$) and $(t_\infty^r)^\gamma = \beta_5$, where

$$\beta_5 = \left\{ a = (a_k) \in \omega : \sup_{n \in \mathbb{N}} \sum_k |d_{nk}^r| < \infty \right\} \quad (1 < p < \infty)$$

Proof. Let $a = (a_k) \in \beta_1$ and $x = (x_k) \in t_p^r$. Then, we obtain by applying the Hölder's inequality that

$$\begin{aligned} \left| \sum_{k=0}^n a_k x_k \right| &= \left| \sum_{k=0}^n \left[\sum_{j=k}^{\infty} \binom{k}{j} (-r)^{k-j} (1-r)^{-(k+1)} y_j \right] a_k \right| \\ &= \left| \sum_{k=0}^n d_{nk}^r y_k \right| \leq \left(\sum_{k=0}^n |d_{nk}^r|^q \right)^{1/q} \left(\sum_{k=0}^n |y_k|^p \right)^{1/p} \end{aligned}$$

where d_{nk}^r is defined by (2.8). Taking supremum over $n \in \mathbb{N}$, we have

$$\begin{aligned} \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^n a_k x_k \right| &\leq \sup_{n \in \mathbb{N}} \left[\left(\sum_{k=0}^n |d_{nk}^r|^q \right)^{1/q} \left(\sum_{k=0}^n |y_k|^p \right)^{1/p} \right] \\ &\leq \|y\|_{\ell_p} \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^n |d_{nk}^r|^q \right)^{1/q} \leq \infty. \end{aligned}$$

This means that $a = (a_k) \in (t_p^r)^\gamma$. Hence $\beta_1 \subset (t_p^r)^\gamma$.

Conversely, let $a = (a_k) \in (t_p^r)^\gamma$ and $x = (x_k) \in t_p^r$. Then one can easily see that $\{\sum_{k=0}^n d_{nk}^r y_k\} \in \ell_\infty$, for all $n \in \mathbb{N}$, whenever $(a_k x_k) \in bs$. This implies that the triangle $D(r) = (d_{nk}^r)$ defined by (2.8), is in the class $(\ell_p : \ell_\infty)$. Hence, the condition $\sup_{n \in \mathbb{N}} \sum_k |d_{nk}^r| < \infty$ is satisfied, which implies that $a = (a_k) \in \beta_1$. That is, $(t_p^r)^\gamma \subset \beta_1$. Therefore, we establish that the γ -dual of t_p^r is the set β_1 .

With the same idea, it can be proved that $(t_1^r)^\gamma = \beta_4$ and $(t_\infty^r)^\gamma = \beta_5$. □

Lemma 2.9. [4, Lemma 5.3] *Let X, Y be any two sequence spaces, A be an infinite matrix and U a triangle matrix. Then, $A \in (X : Y_U)$ if and only if $UA \in (X : Y)$.*

Lemma 2.10. [3, Theorem 3.1] $B^U = (b_{nk})$ be defined via a sequence $a = (a_k) \in \omega$ and inverse of the triangle matrix $U = (u_{nk})$ by

$$b_{nk} = \sum_{j=k}^n a_j v_{jk}$$

for all $k, n \in \mathbb{N}$. Then,

$$\lambda_U^\beta = \{a = (a_k) \in \omega : B^U \in (\lambda : c)\}$$

and

$$\lambda_U^\gamma = \{a = (a_k) \in \omega : B^U \in (\lambda : \ell_\infty)\}.$$

In what follows, for brevity, we write,

$$\bar{a}_{nk} := \sum_{k=0}^n \binom{n}{k} (-r)^{n-k} (1-r)^{-(n+1)} a_{nk}$$

for all $k, m, n \in \mathbb{N}$. From Lemma 2.9 and Lemma 2.10, we have:

Theorem 2.11. *Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $E = (e_{nk})$ are connected with the relation*

$$(2.10) \quad e_{nk} = \bar{a}_{nk}$$

for all $k, n \in \mathbb{N}$ and μ be any given sequence space. Then,

- (i) $A \in (t_p^r : \mu)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in \{t_p^r\}^\beta$ for all $n \in \mathbb{N}$ and $E \in (\ell_p : \mu)$.
- (ii) $A \in (t_\infty^r : \mu)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in \{t_\infty^r\}^\beta$ for all $n \in \mathbb{N}$ and $E \in (\ell_\infty : \mu)$.

Theorem 2.12. *Suppose that the elements of the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation*

$$(2.11) \quad b_{nk} := \sum_{j=n}^{\infty} \binom{j}{n} (1-r)^{n+1} r^{j-n} a_{jk} \quad \text{for all } k, n \in \mathbb{N}.$$

Let μ be any given sequence space. Then,

- (i) $A \in (\mu : t_p^r)$ if and only if $B \in (\mu : \ell_p)$.
- (ii) $A \in (\mu : t_\infty^r)$ if and only if $B \in (\mu : \ell_\infty)$.

The following results were taken from Stieglitz and Tietz [13]:

$$\begin{aligned}
(2.12) \quad & \sum_n a_{nk} \text{ convergent for all } k, \\
(2.13) \quad & \sup_{k,m} \left| \sum_{n=0}^m a_{nk} \right| < \infty, \\
(2.14) \quad & \sup_{n,k} |a_{nk}| < \infty, \\
(2.15) \quad & \sum_n a_{nk} = 0 \text{ for all } k, \\
(2.16) \quad & \lim_m \sum_k \left| \sum_{n=0}^m a_{nk} \right| = \sum_k \left| \sum_n a_{nk} \right|, \\
(2.17) \quad & \lim_m \sum_k \left| \sum_{n=0}^m a_{nk} \right| = 0 \\
(2.18) \quad & \sup_m \sum_k \left| \sum_{n=0}^m a_{nk} \right|^q < \infty \\
(2.19) \quad & \lim_k a_{nk} = 0 \text{ for all } n \\
(2.20) \quad & \sup_n \sum_k |a_{nk} - a_{n,k+1}| < \infty \\
(2.21) \quad & \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} (a_{nk} - a_{n,k+1}) \right|^p < \infty \\
(2.22) \quad & \sup_{K, N \in \mathcal{F}} \left| \sum_{n \in N} \sum_{k \in K} (a_{nk} - a_{n,k+1}) \right| < \infty \\
(2.23) \quad & \sup_n \left| \lim_k a_{nk} \right| < \infty \\
(2.24) \quad & \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} (a_{nk} - a_{n,k-1}) \right|^p < \infty \\
(2.25) \quad & \sup_{K, N \in \mathcal{F}} \left| \sum_{n \in N} \sum_{k \in K} (a_{nk} - a_{n,k-1}) \right| < \infty \\
(2.26) \quad & \lim_n \sum_k |a_{nk}| = \sum_k \left| \lim_n a_{nk} \right| \\
(2.27) \quad & \lim_n \sum_k |a_{nk}| = 0 \\
(2.28) \quad & \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} a_{nk} \right|^p < \infty
\end{aligned}$$

Lemma 2.13. For the characterization of the class $(X : Y)$ for $\{\ell_\infty, \ell_p, \ell_1, c, c_0, bs, cs, c_0s\}$, we can give the necessary and sufficient conditions from Table 2, Table 3, Table 4 and Table 5 where

4. (2.5) for $q = 1$	5. (2.4), (2.26)	6. (2.27)	7. (2.4) for $\alpha_k = 0$, (2.5)
8. (2.14)	9. (2.4), (2.14)	10. (2.4) for $\alpha_k = 0$, (2.14)	11. (2.28)
12. (2.28) for $p = 1$	13. (2.18)	14. (2.12), (2.18)	15. (2.15), (2.18)
16. (2.13)	17. (2.12), (2.13)	18. (2.13), (2.15)	19. (2.18) for $q = 1$
20. (2.12)	21. (2.17)	22. (2.19), (2.20)	23. (2.19), (2.21)
24. (2.19), (2.22)	25. (2.20), (2.23)	26. (2.24)	27. (2.25)
28. (2.20)	29. (2.21)	30. (2.22)	

To \rightarrow	ℓ_∞	c	c_0
From \downarrow			
ℓ_∞	4.	5.	6.
ℓ_p	3.	2.	7.
ℓ_1	8.	9.	10.

Table 2

To \rightarrow	ℓ_∞	ℓ_p	ℓ_1
From \downarrow			
ℓ_∞	4.	11.	12.
c	4.	11.	12.
c_0	4.	11.	12.

Table 3

To \rightarrow	bs	cs	c_0s
From \downarrow			
ℓ_p	13.	14.	15.
ℓ_1	16.	17.	18.
ℓ_∞	19.	20.	21.

Table 4

To \rightarrow	ℓ_∞	ℓ_p	ℓ_1
From \downarrow			
bs	22.	23.	24.
cs	25.	26.	27.
c_0s	28.	29.	30.

Table 5

Now, using the Theorem 2.11, Theorem 2.12 and Lemma 2.13, we can give the some results:

Let $A = (a_{nk})$ be an infinite matrix. Consider the $X \in \{\ell_\infty, c, c_0, bs, cs, c_0s\}$.

$A \in (t_p^r : X)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in \{t_p^r\}^\beta$ for all $n \in \mathbb{N}$ and the conditions **2.**, **3.**, **7.**, **13.**, **14.**, **15.** in Table 2 and Table 4 hold with \bar{a}_{nk} instead of a_{nk} .

$A \in (t_1^r : X)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in \{t_1^r\}^\beta$ for all $n \in \mathbb{N}$ and the conditions **8.**, **9.**, **10.**, **16.**, **17.**, **18.** in Table 2 and Table 4 hold with \bar{a}_{nk} instead of a_{nk} .

$A \in (t_\infty^r : X)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in \{t_\infty^r\}^\beta$ for all $n \in \mathbb{N}$ and the conditions **4.**, **5.**, **6.**, **19.**, **20.**, **21.** in Table 2 and Table 4 hold with \bar{a}_{nk} instead of a_{nk} .

The relation b_{nk} be defined by (2.11).

$A = (a_{nk}) \in (X : t_\infty^r)$ if and only if the conditions **4.**, **22.**, **25.**, **28.** in Table 3 and Table 5 hold with b_{nk} instead of a_{nk} .

$A = (a_{nk}) \in (X : t_p^r)$ if and only if the conditions **11.**, **23.**, **26.**, **29.** in Table 3 and Table 5 hold with b_{nk} instead of a_{nk} .

$A = (a_{nk}) \in (X : t_1^r)$ if and only if the conditions **12.**, **24.**, **27.**, **30.** in Table 3 and Table 5 hold with b_{nk} instead of a_{nk} .

Thus, we have characterized the matrix classes between the new space and the classical spaces. In above results, some examples are given with the different spaces which are obtained from the triangular matrices(cf. [5]) instead of X .

3. SOME GEOMETRIC PROPERTIES OF THE SPACE t_p^r

In this section, we examine some geometric properties of the space t_p^r . First, we define some geometric properties of the spaces. Let $(X, \|\cdot\|)$ be a normed space and let $S(x)$ and $B(x)$ be the unit sphere and unit ball of X , respectively. Consider *Clarkson's modulus of convexity* (see [6, 7]) defined by

$$\delta_X(\theta) = \inf \left\{ 1 - \frac{\|x - y\|}{2}; x, y \in S(x), \|x - y\| = \theta \right\},$$

where $0 \leq \theta \leq 2$. The inequality $\delta_X(\theta) > 0$ for all $\theta \in [0, 2]$ characterizes the uniformly convex spaces. In [11], Gurarii's modulus of convexity is defined by

$$\beta_X(\theta) = \inf \left\{ 1 - \inf_{\alpha \in [0,1]} \|\alpha x + (1-\alpha)y\|; x, y \in S(x), \|x - y\| = \theta \right\},$$

where $0 \leq \theta \leq 2$. It is easily shown that $\delta_X(\theta) \leq \beta_X(\theta) \leq 2\delta_X(\theta)$ for any $0 \leq \theta \leq 2$. Further, if $0 < \beta_X(\theta) < 1$, then X is uniformly convex, and if $\beta_X(\theta) < 1$, then X is strictly convex.

A Banach space X is said to have the Banach-Saks property if every bounded sequence (x_n) in X admits a sequence (z_n) such that the sequence $t_k(z)$ is convergent in norm in X (see [8]), where

$$t_k(z) = \frac{1}{k+1}(z_0 + z_1 + \cdots + z_k) \text{ for all } k \in \mathbb{N}.$$

Let $1 < p < \infty$. A Banach space is said to have the Banach-Saks type p if every weakly null sequence has a subsequence (x_k) such that for some $C > 0$,

$$\left\| \sum_{k=0}^n x_k \right\| < C(n+1)^{1/p}.$$

A Banach space X is said to have the *weak Banach-Saks property* whenever given any weakly null sequence (x_n) in X and there exists a subsequence (z_n) of (x_n) such that the sequence $\{t_k(z)\}$ strongly converges to zero.

In [9], García-Falset introduced the following coefficient:

$$R(X) = \sup \left\{ \liminf_{n \rightarrow \infty} \|x_n + x\|; (x_n) \subset B(x), x_n \xrightarrow{w} 0 \right\}.$$

Remark 3.1. [10] *A Banach space X with $R(X) < 2$ has a weak fixed point property.*

Theorem 3.2. *The space t_p^r has Banach-Saks type p .*

Proof. Let (ε_n) be a sequence of positive numbers for which $\sum_{n=1}^{\infty} \varepsilon_n \leq 1/2$. Let (x_n) be a weakly null sequence in $B(t_p^r(r))$. Set $s_0 = x_0$ and $s_1 = x_{n_1} = x_1$. Then, there exists $t_1 \in \mathbb{N}$ such that

$$\left\| \sum_{i=t_1+1}^{\infty} s_1(i)e^{(i)} \right\|_{t_p^r} < \varepsilon_1.$$

The assumption " (x_n) be a weakly null sequence" implies that $x_n \rightarrow 0$ with respect to the coordinatewise, there exists an $n_2 \in \mathbb{N}$ such that

$$\left\| \sum_{i=0}^{t_1} x_n(i)e^{(i)} \right\|_{t_p^r} < \varepsilon_1,$$

where $n \geq n_2$. Set $s_2 = x_{n_2}$. Then, there exists $t_2 > t_1$ such that

$$\left\| \sum_{i=t_2+1}^{\infty} s_2(i)e^{(i)} \right\|_{t_p^r} < \varepsilon_2.$$

By using the fact that $x_n \rightarrow 0$ with respect to coordinatewise, there exists $n_3 > n_2$ such that

$$\left\| \sum_{i=0}^{t_2} x_n(i)e^{(i)} \right\|_{t_p^r} < \varepsilon_2,$$

where $n \geq n_3$. If we continue this process, we can find two increasing sequences (t_i) and (n_i) of natural numbers such that

$$\left\| \sum_{i=0}^{t_j} x_n(i)e^{(i)} \right\|_{t_p^r} < \varepsilon_j$$

for each $n \geq n_{j+1}$ and

$$\left\| \sum_{i=t_j+1}^{\infty} s_j(i)e^{(i)} \right\|_{t_p^r} < \varepsilon_j,$$

where $s_j = x_{n_j}$. Hence,

$$\begin{aligned}
\left\| \sum_{j=0}^n s_j \right\|_{t_p^r} &= \left\| \sum_{j=0}^n \left(\sum_{i=0}^{t_j-1} s_j(i) e^{(i)} + \sum_{i=t_{j-1}}^{t_j} s_j(i) e^{(i)} + \sum_{i=t_j+1}^{\infty} s_j(i) e^{(i)} \right) \right\|_{t_p^r} \\
&\leq \left\| \sum_{j=0}^n \sum_{i=0}^{t_j-1} s_j(i) e^{(i)} \right\|_{t_p^r} + \left\| \sum_{j=0}^n \sum_{i=t_{j-1}}^{t_j} s_j(i) e^{(i)} \right\|_{t_p^r} \\
&\quad + \left\| \sum_{j=0}^n \sum_{i=t_j+1}^{\infty} s_j(i) e^{(i)} \right\|_{t_p^r} \\
&\leq \left\| \sum_{j=0}^n \left(\sum_{i=t_{j-1}+1}^{t_j} s_j(i) e^{(i)} \right) \right\|_{t_p^r} + 2 \sum_{j=0}^n \varepsilon_j.
\end{aligned}$$

In other respects, one can see that $\|x\|_{t_p^r} < 1$. Thus, $\|x\|_{t_p^r}^p < 1$ and we get

$$\begin{aligned}
\left\| \sum_{j=0}^n \sum_{i=t_{j-1}+1}^{t_j} s_j(i) e^{(i)} \right\|_{t_p^r}^p &= \sum_{j=0}^n \sum_{i=t_{j-1}+1}^{t_j} \left| \sum_{k=i}^{\infty} \binom{k}{i} (1-r)^{i+1} r^{k-i} x_j(k) \right|^p \\
&\leq \sum_{j=0}^n \sum_{i=0}^{\infty} \left| \sum_{k=i}^{\infty} \binom{k}{i} (1-r)^{i+1} r^{k-i} x_j(k) \right|^p \\
&\leq (n+1).
\end{aligned}$$

Hence, we get

$$\left\| \sum_{j=0}^n \sum_{i=t_{j-1}+1}^{t_j} s_j(i) e^{(i)} \right\|_{t_p^r} \leq (n+1)^{1/p}.$$

By using the inequality $1 \leq (n+1)^{1/p}$ for all $n \in \mathbb{N}$ and $1 \leq p < \infty$, we obtain

$$\left\| \sum_{j=0}^n s_j \right\|_{t_p^r} \leq (n+1)^{1/p} + 1 \leq 2(n+1)^{1/p}.$$

Thus, the space t_p^r has Banach-Saks type p . □

Remark 3.3. Note that $R(t_p^r) = R(\ell_p) = 2^{1/p}$, since t_p^r is an isomorphic copy of the ℓ_p .

Hence, by Remarks 3.1 and 3.3, we have the following:

Corollary 3.4. Let $1 < p < \infty$. Then, the space t_p^r has the weak fixed point property.

Theorem 3.5. Gurarii's modulus of convexity for the normed space t_p^r is

$$\beta_{t_p^r}(\theta) \leq 1 - \left[1 - \left(\frac{\theta}{2} \right)^p \right]^{1/p},$$

where $0 \leq \theta \leq 2$.

Proof. Let $x \in t_p^r$. Then, we obtain

$$\|x\|_{t_p^r} = \|T(r)x\|_p = \left[\sum_n |(T(r)x)_n|^p \right]^{1/p}.$$

Let $0 \leq \theta \leq 2$ and take into consideration this sequences

$$\begin{aligned}
x = (x_k) &= \left\{ T[-r/(1-r)] \left[\left[1 - \left(\frac{\theta}{2} \right)^p \right]^{1/p} \right], T[-r/(1-r)] \left(\frac{\theta}{2} \right), 0, 0, \dots \right\}, \\
y = (y_k) &= \left\{ T[-r/(1-r)] \left[\left[1 - \left(\frac{\theta}{2} \right)^p \right]^{1/p} \right], T[-r/(1-r)] \left(-\frac{\theta}{2} \right), 0, 0, \dots \right\}.
\end{aligned}$$

Because of $a_k = [T(r)x]_k$ and $b_k = [T(r)y]_k$, we get

$$\begin{aligned} a = (a_k) &= \left\{ \left[1 - \left(\frac{\theta}{2} \right)^p \right]^{1/p}, \frac{\theta}{2}, 0, 0, \dots \right\}, \\ b = (b_k) &= \left\{ \left[1 - \left(\frac{\theta}{2} \right)^p \right]^{1/p}, -\frac{\theta}{2}, 0, 0, \dots \right\}. \end{aligned}$$

By using the sequences $x = (x_k)$ and $y = (y_k)$, we obtain following equalities

$$\begin{aligned} \|x\|_{t_p^r}^p = \|T(r)x\|_p &= \left| \left[1 - \left(\frac{\theta}{2} \right)^p \right]^{1/p} \right|^p + \left| -\frac{\theta}{2} \right|^p \\ &= 1 - \left(\frac{\theta}{2} \right)^p + \left(\frac{\theta}{2} \right)^p = 1, \end{aligned}$$

$$\begin{aligned} \|y\|_{t_p^r}^p = \|T(r)y\|_p &= \left| \left[1 - \left(\frac{\theta}{2} \right)^p \right]^{1/p} \right|^p + \left| -\frac{\theta}{2} \right|^p \\ &= 1 - \left(\frac{\theta}{2} \right)^p + \left(\frac{\theta}{2} \right)^p = 1 \end{aligned}$$

and

$$\|x - y\|_{t_p^r}^p = \|T(r)x - T(r)y\|_p = \left\{ \left| \left[1 - \left(\frac{\theta}{2} \right)^p \right]^{1/p} - \left[1 - \left(\frac{\theta}{2} \right)^p \right]^{1/p} \right|^p + \left| \frac{\theta}{2} - \left(-\frac{\theta}{2} \right) \right|^p \right\}^{1/p} = \theta.$$

For $0 \leq \alpha \leq 1$

$$\begin{aligned} \inf_{\alpha \in [0,1]} \|\alpha x + (1-\alpha)y\|_{t_p^r} &= \inf_{\alpha \in [0,1]} \|\alpha T(r)x + (1-\alpha)T(r)y\|_p \\ &= \inf_{\alpha \in [0,1]} \left\{ \left| \alpha \left[1 - \left(\frac{\theta}{2} \right)^p \right]^{1/p} + (1-\alpha) \left[1 - \left(\frac{\theta}{2} \right)^p \right]^{1/p} \right|^p + \left| \alpha \left(\frac{\theta}{2} \right)^p + (1-\alpha) \left(-\frac{\theta}{2} \right)^p \right|^p \right\}^{1/p} \\ &= \inf_{\alpha \in [0,1]} \left[1 - \left(\frac{\theta}{2} \right)^p + (2\alpha - 1)^p \left(\frac{\theta}{2} \right)^p \right]^{1/p} \\ &= \left[1 - \left(\frac{\theta}{2} \right)^p \right]^{1/p}. \end{aligned}$$

Therefore, for $1 \leq p < \infty$, we have

$$\beta_{t_p^r}(\theta) \leq 1 - \left[1 - \left(\frac{\theta}{2} \right)^p \right]^{1/p}.$$

The completes the proof. \square

Corollary 3.6. *The following statements hold:*

- (i) For $\theta > 2$, $\beta_{t_p^r}(\theta) = 1$. Thus, t_p^r is strictly convex.
- (ii) For $0 < \theta \leq 2$, $\beta_{t_p^r}(\theta) \leq 1$. Thus, t_p^r is uniformly convex.

Corollary 3.7. For $\alpha = 1/2$, $\beta_{t_p^r}(\theta) = \delta_{t_p^r}(\theta)$.

4. CONCLUSION

The purpose of this paper is twohold. In section 2, it was obtained the new matrix domain with the well known classical sequence spaces and an infinite matrix. We use the Taylor method as an infinite matrix for the matrix domain, in this study. The Taylor method which known then as the circle method of order r ($0 < r < 1$).

It is well known that every Banach space is isomorphic to a subspace of a Banach space and every separable reflexive Banach space is isomorphic to a subspace of a separable reflexive space with Schauder basis. In section 2, we construct new sequence spaces, which naturally emerge from the classical sequence spaces and Taylor transform. The new space t_p^r obtained as domain of Taylor upper triangle matrix in the space ℓ_p ($1 \leq p \leq \infty$). In this paper, since the new space t_p^r is isomorphic copy to the space ℓ_p , the space t_p^r has a Schauder basis and

also some topological properties of new constructed space are similar to the classical space ℓ_p . Then, some basic results of the space t_p^r such as absolute property, BK -space, isomorphism, Schauder basis, separability and some inclusion relations are given but not proved because of natural continuation of the space ℓ_p . Therefore, we show that the space t_p^r is a Hilbert space for $p = 2$. Also, the proofs of dual spaces theorems are given which are important. The characterized matrix classes between the new space and the classical spaces are shown in the table form. Further, based on the tables, some results of matrix classes are given as examples.

In section 3, we focus on some important results which are geometric properties of Banach spaces. In Banach Space Theory, geometric properties are play a crucial role. As is well known, among all infinite dimensional Banach spaces, Hilbert spaces have the nicest geometric properties. We present the some geometric properties of the space t_p^r such as Clarkson's modulus of convexity, Gurarii's modulus of convexity, Banach-Saks property, weak Banach-Saks property, weak fixed point property, strictly convexity, uniform convexity.

The geometric properties in this study will form the basis for future works. Then, the aim of this paper is to present an in-depth and up to date coverage of the main ideas, concepts and most important results related to sequence spaces and geometric properties of Banach Space.

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